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Conjugate Temperatures

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We develop Cauchy-Riemann equations for pairs of temperature functions with boundary values in $L^1(R, dx/(1+x^2))$.

1. INTRODUCTION

The Cauchy–Riemann equations $D_x u = D_y v$ and $D_y u = -D_x v$ can be viewed as a splitting of Laplace's equation $D_x^2 u + D_y^2 u = 0$. The pair of solutions of the Cauchy–Riemann equations, for a large class of functions, is related via the Hilbert transform.

We show that for a large class of functions the Hilbert transform similarly splits the heat equation $D_t u(x, t) = D_x^2 u(x, t)$.

Write $(u(x, t), v(x, t)) \in \mathscr{A}\mathscr{H}$ if

$$D_x u(x, t) = -iD_t^{1/2} v(x, t) \tag{1}$$

and

$$iD_t^{1/2} u(x, t) = D_x v(x, t) \tag{2}$$

for $t > 0$ and $x \in R$, where $D_t^{1/2}$ is a Weyl fractional derivative operator (see below).

We show that for $g \in L^1(R, dx/(1+x^2))$, if $u(x, t) = g * k(x, t)$, where $k(x, t)$ is the Gauss–Weierstrass kernel, then $(u(x, t), \mathscr{H}u(x, t)) \in \mathscr{A}\mathscr{H}$, where $\mathscr{H}u(x, t)$ denotes the Hilbert transform of $u(x, t)$ with respect to the first variable.

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Weyl's fractional integral of order $\alpha > 0$ is defined by

$$D^{-\alpha}f(t) = \frac{e^{i\pi\alpha}}{\Gamma(\alpha)} \int_t^\infty f(u)(u-t)^{\alpha-1}\,du \tag{3}$$

and the fractional derivative of order $\alpha > 0$ is

$$D^{\alpha}f(t) = \frac{e^{i\pi\tilde{\alpha}}}{\Gamma(\tilde{\alpha})} \int_t^\infty f^{(n)}(u)(u-t)^{\tilde{\alpha}-1}\,du, \tag{4}$$

where $\alpha = n - \tilde{\alpha}$, for a positive integer n and $0 < \tilde{\alpha} \leqslant 1$.

This version of the Weyl fractional derivative was proposed by M. Riesz in [3]. It is shown in [3] that for functions f which are sufficiently regular we have $D^\alpha D^\beta f = D^{\alpha+\beta} f$ and $(d/dt)\, D^{-1} f = f$. In particular, if $u(x, t)$ and $v(x, t)$ are nice enough, then $(u(x, t), v(x, t)) \in \mathscr{A}\mathscr{H}$ implies

$$\begin{aligned} D_t u(x, t) &= D_t^{1/2} D_t^{1/2} u(x, t) = -i D_t^{1/2} D_x v(x, t) \\ &= -i D_x D_t^{1/2} v(x, t) = D_x^2 u(x, t) \end{aligned}$$

and

$$\begin{aligned} D_t v(x, t) &= D_t^{1/2} D_t^{1/2} v(x, t) = i D_t^{1/2} D_x u(x, t) \\ &= i D_x D_t^{1/2} u(x, t) = D_x^2 v(x, t) \end{aligned}$$

so that $u(x, t)$ and $v(x, t)$ satisfy the heat equation.

For $f \in L^p(R)$, $1 \leqslant p < \infty$, the Hilbert transform is defined a.e. by

$$\mathscr{H}f(x) = \text{p.v.} \frac{1}{\pi} \int_R \frac{f(s)}{x-s}\,ds. \tag{5}$$

We prove our results for $f \in L^1(R, dx/(1+x^2))$; this space contains $BMO(R)$. For $f \in L^1(R, dx/(1+x^2))$, the above integral might fail to converge. In this case the Hilbert transform may be defined a.e. up to additive constants by

$$\mathscr{H}f(x) = \text{p.v.} \frac{1}{\pi} \int_R \left(\frac{1}{x-s} + \frac{1}{(s)_1} \right) f(s)\,ds, \tag{6}$$

where $1/(s)_\delta = 1/s$ for $|s| > \delta$ and zero otherwise.

Suppose $f \in L^2(R)$. The Fourier transform of f is defined

$$\hat{f}(t) = \int_R f(x)\, e^{-ixt}\,dx. \tag{7}$$

The function f is obtained from its Fourier transform by the Fourier inversion formula

$$f(x) = \frac{1}{2\pi} \int_R \hat{f}(t)\, e^{ixt}\, dt. \tag{8}$$

The Fourier transforms of $f \in L^2(R)$ and of $\mathscr{H}f$ are related by the identity

$$\widehat{\mathscr{H}f}(x) = -i \operatorname{sgn} x \cdot \hat{f}(x). \tag{9}$$

2. Conjugate Temperatures

Let $\mathscr{G}(x) = (1/\sqrt{2\pi})\, e^{-x^2/2}$. The fundamental solution of the heat equation is the Gauss–Weierstrass kernel

$$k(x, t) = \frac{1}{\sqrt{2t}}\, \mathscr{G}\left(\frac{x}{\sqrt{2t}}\right) = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}. \tag{10}$$

We define its conjugate by

$$S(x, t) = \frac{1}{\sqrt{2t}}\, S\left(\frac{x}{\sqrt{2t}}\right), \tag{11}$$

where

$$S(x) = \mathscr{H}\mathscr{G}(x) = \frac{1}{\pi}\, e^{-x^2/2} \int_0^x e^{u^2/2}\, du. \tag{12}$$

See [1].

Since $\hat{k}(x, t) = e^{-tx^2}$ for $t > 0$, we have from the Fourier inversion formula

$$k(x, t) = \frac{1}{\pi} \int_0^\infty e^{-ty^2} \cos xy\, dy \tag{13}$$

and

$$S(x, t) = \frac{1}{\pi} \int_0^\infty e^{-ty^2} \sin xy\, dy. \tag{14}$$

Theorem 1. *For $t > 0$ and $x \in R$,*

$$(k(x, t), S(x, t)) \in \mathscr{A}\mathscr{H}. \tag{15}$$

Proof. Since

$$D_t S(x, t) = \frac{-1}{\pi} \int_0^\infty y^2 e^{-ty^2} \sin xy \, dy,$$

we have

$$\begin{aligned} -iD_t^{1/2} S(x, t) &= \frac{1}{\sqrt{\pi}} \int_0^\infty D_t S(x, u+t)\, u^{-1/2} \, du \\ &= \frac{-1}{\pi\sqrt{\pi}} \int_0^\infty \left(\int_0^\infty y^2 e^{-(u+t)y^2} \sin xy \, dy \right) u^{-1/2} \, du \\ &= \frac{-1}{\pi\sqrt{\pi}} \int_0^\infty \left(\int_0^\infty e^{-uy^2} u^{-1/2} \, du \right) y^2 e^{-ty^2} \sin xy \, dy \\ &= -\frac{1}{\pi} \int_0^\infty y e^{-ty^2} \sin xy \, dy = D_x k(x, t). \end{aligned}$$

Similarly,

$$\begin{aligned} iD_t^{1/2} k(x, t) &= \frac{-1}{\sqrt{\pi}} \int_0^\infty D_t k(x, u+t)\, u^{-1/2} \, du \\ &= \frac{1}{\pi\sqrt{\pi}} \int_0^\infty \left(\int_0^\infty y^2 e^{-(u+t)y^2} \cos xy \, dy \right) u^{-1/2} \, du \\ &= \frac{1}{\pi\sqrt{\pi}} \int_0^\infty \left(\int_0^\infty e^{-uy^2} u^{-1/2} \, du \right) y^2 e^{-ty^2} \cos xy \, dy \\ &= \frac{1}{\pi} \int_0^\infty y e^{-ty^2} \cos xy \, dy = D_x S(x, t). \quad \blacksquare \end{aligned}$$

We now consider conjugate temperatures which are convolutions of initial values $g(x) \in L^1(R, dx/(1+x^2))$ with $k(x, t)$ and $S(x, t)$. However, for $g \in L^1(R, dx/(1+x^2))$, the convolution $g * S(x, t)$ is not always defined because $S(x, t) = O(1/|x|)$ as $|x| \to \infty$; see, e.g., [2]. Therefore, in analogy with (6), we define up to additive constants

$$\tilde{S}g(x, t) = \int_R [S(x-y, t) + S(y)]\, g(y)\, dy. \tag{16}$$

This operator is defined for $g \in L^1(R, dx/(1+x^2))$ since, as we will show later,

$$S(x-y, t) + S(y) = O\left(\frac{1}{y^2}\right), \qquad |y| \to \infty.$$

We show that for $g \in L^1(R, dx/(1+x^2))$ we have

$$(g * k(x, t),\ \tilde{S}g(x, t)) \in \mathscr{A}\mathscr{H}.$$

For $g \in L^p(R)$, $1 \leqslant p < \infty$, since $D_t^{1/2}(1) = 0$, this is equivalent to

$$(g * k(x, t), g * S(x, t)) \in \mathscr{A}\mathscr{H}.$$

We denote by C a positive constant, not necessarily the same on different occurrences.

LEMMA 2. *If* $0 < \alpha < 1$, $\beta > 2\alpha$, $x > 0$; *and if for all* $t > 0$,

$$|D_t w(t)| \leqslant \min\left\{\frac{1}{x^\beta}, \frac{1}{t^{\beta/2}}\right\},$$

then

$$|D_t^{1-\alpha} w(t)| \leqslant C \cdot \min\left\{\frac{1}{x^{\beta-2\alpha}}, \frac{1}{t^{(\beta-2\alpha)/2}}\right\}. \tag{17}$$

Proof. Since

$$D_t^{1-\alpha} w(t) = \frac{e^{i\pi\alpha}}{\Gamma(\alpha)} \int_0^\infty D_t w(u+t)\, u^{\alpha-1}\, du,$$

we have

$$|D_t^{1-\alpha} w(t)| \leqslant \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{1}{(u+t)^{\beta/2}} u^{\alpha-1}\, du = \frac{\Gamma((\beta-2\alpha)/2)}{\Gamma(\beta/2)} \cdot \frac{1}{t^{(\beta-2\alpha)/2}}.$$

We also have

$$\begin{aligned} |D_t^{1-\alpha} w(t)| &\leqslant \frac{1}{\Gamma(\alpha)} \left(\frac{1}{x^\beta} \int_0^{x^2} u^{\alpha-1}\, du + \int_{x^2}^\infty \frac{1}{(u+t)^{\beta/2}} u^{\alpha-1}\, du \right) \\ &\leqslant \frac{1}{\Gamma(\alpha)} \left(\frac{1}{\alpha x^{\beta-2\alpha}} + \frac{2}{\beta-2\alpha} \cdot \frac{1}{x^{\beta-2\alpha}} \right). \quad \blacksquare \end{aligned}$$

LEMMA 3. *For* $t > 0$ *and* $x \in R$, *we have*

$$|D_t k(x, t)| \leqslant C \cdot \min\left\{\frac{1}{|x|^3}, \frac{1}{t^{3/2}}\right\}, \tag{18}$$

$$|D_x S(x, t)| \leqslant C \cdot \min\left\{\frac{1}{x^2}, \frac{1}{t}\right\}, \tag{19}$$

and

$$|D_t S(x, t)| \leqslant C \cdot \min\left\{\frac{1}{|x|^3}, \frac{1}{t^{3/2}}\right\}. \tag{20}$$

Proof. Note that

$$D_t k(x, t) = \frac{1}{4\sqrt{\pi}}\left(\frac{x^2}{2t} - 1\right)\frac{1}{t^{3/2}} e^{-x^2/4t}$$

and for t, $\beta > 0$

$$\frac{1}{t^\beta} e^{-x^2/4t} \leqslant \left(\frac{4\beta}{e}\right)^\beta \cdot \frac{1}{|x|^{2\beta}}. \tag{21}$$

We have for $x^2 > 2t$

$$\begin{aligned}|D_t k(x, t)| &\leqslant \frac{1}{4\sqrt{\pi}} \cdot \frac{x^2}{2} \cdot \frac{1}{t^{5/2}} e^{-x^2/4t} \\ &\leqslant \frac{x^2}{8\sqrt{\pi}} \cdot \left(\frac{10}{e}\right)^{5/2} \cdot \frac{1}{|x|^5} \\ &= \frac{1}{8\sqrt{\pi}}\left(\frac{10}{e}\right)^{5/2} \cdot \min\left\{\frac{1}{|x|^3}, \frac{1}{(2t)^{3/2}}\right\}.\end{aligned}$$

We have for $x^2 \leqslant 2t$

$$|D_t k(x, t)| \leqslant \frac{1}{4\sqrt{\pi}} \frac{1}{t^{3/2}} = \frac{1}{4\sqrt{\pi}} \cdot \min\left\{\frac{2^{3/2}}{|x|^3}, \frac{1}{t^{3/2}}\right\}.$$

This proves (18).

Observe that $D_x S(x, t) = i D_t^{1/2} k(x, t)$. Therefore by Lemma 2 applied to (18), we have

$$|D_x S(x, t)| \leqslant C \cdot \min\left\{\frac{1}{x^2}, \frac{1}{t}\right\},$$

which proves (19).

Since $D_t S(x, t) = D_t^{1/2} D_t^{1/2} S(x, t) = i D_t^{1/2}(D_x k(x, t))$, the proof of (20) is similar. ∎

LEMMA 4. *For $t > 0$ and $x, y \in R$,*

$$|S(x-y, t) + S(y)| \leqslant C \cdot \left(\frac{|x|}{t} \chi_{\{|y| \leqslant 2|x|\}}(y) + \frac{|x|}{y^2} \chi_{\{|y| > 2|x|\}}(y)\right) + C \cdot \min\left\{\frac{|t - 1/2|}{|y|^3}, \frac{|\sqrt{2t} - 1|}{\sqrt{t}}\right\}. \tag{22}$$

Proof. Since $S(y, t) = -S(-y, t)$ we have

$$|S(y, t) + S(x-y, t)| \leqslant \operatorname{sgn} x \int_{y-x}^{y} |D_s S(s, t)| \, ds$$

$$\leqslant C \cdot \operatorname{sgn} x \int_{y-x}^{y} \min\left\{\frac{1}{t}, \frac{1}{s^2}\right\} ds.$$

For all y, the last term is majorized by $C \cdot |x|/t$. If $|y| > 2|x|$, then the integral is majorized by

$$\operatorname{sgn} x \int_{y-x}^{y} \frac{1}{s^2} \, ds = \left|\frac{x}{y(y-x)}\right| \leqslant \frac{2|x|}{y^2}.$$

We have shown that

$$|S(y, t) + S(x-y, t)| \leqslant C \cdot \left(\frac{|x|}{t} \chi_{\{|y| \leqslant 2|x|\}}(y) + \frac{|x|}{y^2} \chi_{\{|y| > 2|x|\}}(y)\right). \tag{23}$$

We show next that

$$|S(y, t) - S(y)| \leqslant C \cdot \min\left\{\frac{|t-1/2|}{|y|^3}, \frac{|\sqrt{2t}-1|}{\sqrt{t}}\right\}. \tag{24}$$

Since $S(y) = S(y, \frac{1}{2})$, we have

$$|S(y, t) - S(y)| \leqslant \operatorname{sgn}\left(t - \frac{1}{2}\right) \int_{1/2}^{t} |D_\tau S(y, \tau)| \, d\tau$$

$$\leqslant C \cdot \operatorname{sgn}\left(t - \frac{1}{2}\right) \int_{1/2}^{t} \min\left\{\frac{1}{|y|^3}, \frac{1}{\tau^{3/2}}\right\} d\tau$$

$$\leqslant C \cdot \min\left\{\frac{|t-1/2|}{|y|^3}, \frac{|\sqrt{2t}-1|}{\sqrt{t}}\right\}.$$

By combining (23) and (24) we get the result. ∎

In particular Lemma 4 shows

$$|S(x-y, t) + S(y)| = O\left(\frac{1}{y^2}\right), \qquad |y| \to \infty. \tag{25}$$

THEOREM 5. *If $g(x) \in L^1(R, dx/(1+x^2))$, then for $t > 0$ and $x \in R$,*

$$-iD_t^{1/2} \tilde{S}g(x, t) = D_x(g * k)(x, t). \tag{26}$$

Proof. It is easy to see that

$$D_t \int_R (S(x-y, t) + S(y))\, g(y)\, dy = \int_R D_t S(x-y, t)\, g(y)\, dy.$$

Therefore,

$$\begin{aligned}
D_t^{1/2} &\int_R (S(x-y, t) + S(y))\, g(y)\, dy \\
&= \frac{i}{\sqrt{\pi}} \int_0^\infty D_t \left(\int_R (S(x-y, u+t) + S(y))\, g(y)\, dy \right) u^{-1/2}\, du \\
&= \frac{i}{\sqrt{\pi}} \int_0^\infty \left(\int_R D_t S(x-y, u+t)\, g(y)\, dy \right) u^{-1/2}\, du \\
&= \frac{i}{\sqrt{\pi}} \int_R g(y) \left(\int_0^\infty D_t S(x-y, u+t)\, u^{-1/2}\, du \right) dy \\
&= \int_R g(y)\, D_t^{1/2}\, S(x-y, t)\, dy \\
&= i \int_R g(y)\, D_x k(x-y, t)\, dy \\
&= i D_x (g * k)(x, t).
\end{aligned}$$

Here the application of Fubini's theorem is justified by

$$\begin{aligned}
\int_0^\infty |D_t S(y, u+t)|\, u^{-1/2}\, du &\leqslant C \int_0^\infty \min\left\{ \frac{1}{|y|^3}, \frac{1}{(u+t)^{3/2}} \right\} u^{-1/2}\, du \\
&\leqslant C \cdot \min\left\{ \frac{1}{y^2}, \frac{1}{t} \right\}. \quad \blacksquare
\end{aligned}$$

THEOREM 6. *If* $g(x) \in L^1(R, dx/(1+x^2))$, *then for* $t > 0$ *and* $x \in R$,

$$i D_t^{1/2} (g * k)(x, t) = D_x \tilde{S} g(x, t). \tag{27}$$

Proof.

$$\begin{aligned}
D_t^{1/2} &\int_R g(x-y)\, k(y, t)\, dy \\
&= \frac{i}{\sqrt{\pi}} \int_0^\infty D_t \left(\int_R g(x-y)\, k(y, u+t)\, dy \right) u^{-1/2}\, du
\end{aligned}$$

$$= \frac{i}{\sqrt{\pi}} \int_0^{\infty} \left(\int_R g(x-y)\, D_t k(y, u+t)\, dy \right) u^{-1/2}\, du$$

$$= \frac{i}{\sqrt{\pi}} \int_R g(x-y) \left(\int_0^{\infty} D_t k(y, u+t)\, u^{-1/2}\, du \right) dy$$

$$= \int_R g(x-y)\, D_t^{1/2}\, k(y, t)\, dy$$

$$= -i \int_R g(x-y)\, D_y S(y, t)\, dy$$

$$= -i \int_R g(y)\, D_y S(x-y, t)\, dy$$

$$= -i \int_R g(y)\, D_x (S(x-y, t) + S(y))\, dy$$

$$= -i D_x \tilde{S} g(x, t).$$

Here the application of Fubini's theorem is justified by (18). ∎

It is well-known that for $g \in L^1(R, dx/(1+x^2))$,

$$\lim_{t \to 0+} g * k(x, t) = g(x)$$

at all Lebesgue points of $g(x)$. (In fact this holds for a much larger class of functions; see, e.g., [4].) The following theorem was proved in [1] for $g \in L^p(R)$, $1 \leqslant p < \infty$.

THEOREM 7. *For* $g \in L^1(R, dx/(1+x^2))$,

$$\lim_{t \to 0+} \tilde{S} g(x, t) = \mathscr{H} g(x) \qquad \text{a.e. on } R. \tag{28}$$

Proof.

$$\tilde{S} g(x, t) = \int_R (S(x-y, t) + S(y))\, g(y)\, dy$$

$$= \int_R \left(S(x-y, t) - \frac{1}{\pi (x-y)_{\sqrt{2t}}} \right) g(y)\, dy$$

$$+ \int_R \left(S(y) - \frac{1}{\pi (y)_1} \right) g(y)\, dy$$

$$+ \frac{1}{\pi} \int_R \left(\frac{1}{(x-y)_{\sqrt{2t}}} + \frac{1}{(y)_1} \right) g(y)\, dy.$$

The second integral converges since

$$\left| S(y) - \frac{1}{\pi(y)_1} \right| \leqslant \frac{C}{|y|^3}, \tag{29}$$

see [1]. Since $\tilde{S}$ and $\mathscr{H}$ are defined up to additive constants for $g \in L^1(R, dx/(1+x^2))$, we may ignore this term. The third integral converges to $\mathscr{H}g(x)$ a.e. on R.

It is enough to show therefore that the first integral converges to zero at all Lebesgue points of g. Fix $a > 1$. We have

$$\begin{aligned}
\int_R \left(S(x-y, t) - \frac{1}{\pi(x-y)_{\sqrt{2t}}} \right) g(y)\, dy
&= \int_{|y| \leqslant a\sqrt{2t}} \left(S(y, t) - \frac{1}{\pi(y)_{\sqrt{2t}}} \right) g(x-y)\, dy \\
&\quad + \int_{|y| > a\sqrt{2t}} \left(S(y, t) - \frac{1}{\pi(y)_{\sqrt{2t}}} \right) g(x-y)\, dy \\
&= I_1 + I_2.
\end{aligned}$$

Since $S(y, t)$ and $1/\pi(y)_{\sqrt{2t}}$ are odd functions,

$$I_1 = \int_{|y| \leqslant a\sqrt{2t}} \left(S(y, t) - \frac{1}{\pi(y)_{\sqrt{2t}}} \right) (g(x-y) - g(x))\, dy.$$

Since $S(x) = O(1/|x|)$ as $|x| \to \infty$, $S(x)$ is a bounded function, so $|S(y, t)| \leqslant C/\sqrt{2t}$. Clearly also $|1/\pi(y)_{\sqrt{2t}}| \leqslant 1/(\pi\sqrt{2t})$, so that

$$|I_1| \leqslant \frac{C}{\sqrt{2t}} \int_{|y| \leqslant a\sqrt{2t}} |g(x-y) - g(x)|\, dy \to 0 \tag{30}$$

as $t \to 0^+$, at all Lebesgue points of g.

Since

$$S(y, t) = \frac{1}{\sqrt{2t}}\, S\left(\frac{y}{\sqrt{2t}} \right),$$

we have from (29)

$$\left| S(y, t) - \frac{1}{\pi(y)_{\sqrt{2t}}} \right| \leqslant \frac{Ct}{|y|^3}.$$

Thus

$$|I_2| \leqslant Ct \int_{|y| > a\sqrt{2t}} \frac{|g(x-y)|}{|y|^3}\, dy.$$

Now decompose $g=g_1+g_2$, where $g_1(x-y)=0$ for $|y|>1$ and $g_2(x-y)=0$ for $|y|\leqslant 1$. We have

$$|I_2|\leqslant Ct\int_{a\sqrt{2t}<|y|<1}\frac{|g_1(x-y)|}{|y|^3}\,dy+Ct\int_{|y|>1}\frac{|g_2(x-y)|}{|y|^3}\,dy. \tag{31}$$

Note that

$$\int_{|y|>1}\frac{|g_2(x-y)|}{|y|^3}\,dy=C(x)<\infty.$$

Let $\delta>0$, and observe that

$$\begin{aligned}\delta^2\int_{|y|>\delta}\frac{|g_1(x-y)|}{|y|^3}\,dy&=\delta^2\sum_{l=0}^{\infty}\int_{2^l\delta<|y|\leqslant 2^{l+1}\delta}\frac{|g_1(x-y)|}{|y|^3}\,dy\\&\leqslant\delta^2\sum_{l=0}^{\infty}\frac{1}{2^{3l}\delta^3}\int_{|y|\leqslant 2^{l+1}\delta}|g_1(x-y)|\,dy\\&\leqslant C\sum_{l=0}^{\infty}\frac{1}{2^{2l}}\mathscr{M}g_1(x)\\&\leqslant C\mathscr{M}g_1(x),\end{aligned}$$

where $\mathscr{M}g_1$ is the Hardy–Littlewood maximal function. Choosing $\delta=a\sqrt{2t}$ gives

$$t\int_{|y|>a\sqrt{2t}}\frac{|g_1(x-y)|}{|y|^3}\,dy\leqslant\frac{C}{a^2}\mathscr{M}g_1(x), \tag{32}$$

so that by combining (30), (31), and (32) we have

$$\limsup_{t\to 0+}\left|\int_R\left(S(x-y,t)-\frac{1}{\pi(x-y)_{\sqrt{2t}}}\right)g(y)\,dy\right|\leqslant\frac{C}{a^2}\mathscr{M}g_1(x).$$

Since a can be chosen arbitrarily large, the theorem is proved. ∎

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